Conformal Field Theory and Gravity

Solutions to Problem Set 3 — Fall 2024

1. The Penrose process

(a) The measure is trivially invariant under translations of t and ϕ (it has no t, ϕ dependence), thus K^μ and R^μ are Killing vectors.

(b) Using the metric,

$$0 = g_{\mu\nu} K^\mu K^\mu = -\frac{1}{\rho^2}(\Delta - a^2 \sin^2\theta) \tag{1}$$

Recalling the definition of Δ,

$$K_\mu K^\mu = 0 \implies (r - GM)^2 = G^2 M^2 - a^2 \cos^2\theta \tag{2}$$

(c) Using that at the horizon, $\Delta(r_+) = 0$, $2GMr_+ = r_+^2 + a^2$, we write

$$g_{tt} = -\left(1 - \frac{2GMr}{\rho^2}\right) = -\frac{1}{\rho^2}(\Delta(r) - a^2 \sin^2\theta) \to \frac{a^2 \sin^2\theta}{\rho^2} \tag{3}$$

$$g_{t\phi} \to -\frac{a(r_+^2 + a^2)\sin^2\theta}{\rho^2} \tag{4}$$

$$g_{\phi\phi} \to \frac{(r_+^2 + a^2)\sin^2\theta}{\rho^2} \tag{5}$$

Thus,

$$0 = \chi^\mu \chi_\mu = g_{tt} + 2\Omega_H g_{t\phi} + \Omega_H^2 g_{\phi\phi} = (\Omega_H(r_+^2 + a^2) - a)^2 \tag{6}$$

which implies the given result. Then, the condition $p_2^\mu \chi_\mu < 0$ reads

$$p_2^\mu K_\mu + \Omega_H p_2^\mu R_\mu = -E_2 + \Omega_H L_2 < 0 \implies L_2 < \frac{E_2}{\Omega_H} \tag{7}$$

Since we consider E_2 negative, L_2 is negative. Thus, it reduces to angular momentum of the black hole.

(d) The metric on the spatial slice Σ corresponding $dt = 0$, $dr = 0$ at $r = r_+$ reads,

$$ds_\Sigma^2 = (r_+^2 + a^2\cos^2\theta)d\theta^2 + \left[\frac{(r_+^2 + a^2)^2 \sin^2\theta}{r_+^2 + a^2\cos^2\theta}\right] d\phi^2 \tag{8}$$

The determinant of this metric is

$$|\gamma| = (r_+^2 + a^2)^2 \sin^2\theta \tag{9}$$

Thus, one finds

$$A = \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\pi} d\phi \sqrt{|\gamma|} = 4\pi(r_+^2 + a^2) \tag{10}$$

(e) Plugging $r_+ = GM + \sqrt{G^2M^2 - a^2}$ and using $a = J/M$, one finds

$$A = 8\pi G^2 \left(M^2 + \sqrt{M^4 - (J/G)^2} \right) \tag{11}$$

Taking the variations with respect to M and J,

$$\delta A = \frac{\partial A}{\partial M}\delta M + \frac{\partial A}{\partial J}\delta J \tag{12}$$

one obtains

$$\delta A = \frac{8\pi G a}{\sqrt{G^2M^2 - a^2}}(\Omega_H^{-1}\delta M - \delta J) \tag{13}$$

In the process we considered, after waiting a long time for the p_1 particle to fly to infinity, we can describe the physics by a new Kerr black hole, with its momentum and the black hole mass changed by

$$\delta M = E_2 \quad \delta J = L_2 \tag{14}$$

However, we saw that the amount of energy we can extract is not arbitrary and has to satisfy the bound, which implies $\Omega_H^{-1}\delta M - \delta J > 0$. This precisely ensures that

$$\delta A > 0 \tag{15}$$

as required by the second law.

(f) The key of the process we discussed is that we were allowed to reduce the mass of the black hole, provided that we also reduce the rotation of the black hole. If we want to minimize the mass of the final black hole, we should start a process where we keep A fixed, and we reduce $J \to 0$. In this case we would have that the final mass, called the irreducible mass $M_{\rm irr} = M_{\rm irr}(M, J)$, is

$$8\pi G^2 \left(M^2 + \sqrt{M^4 - (J/G)^2} \right) = 16\pi G^2 M_{\rm irr}^2 \tag{16}$$

In this case, by energy conservation

$$M + E_0 = M_{\rm irr} + E_1 \tag{17}$$

we would have been able to extract

$$E_1 - E_0 = M - M_{\rm irr} \tag{18}$$

2. Radiating Black Holes

(a) $T_H = \frac{1}{8\pi M}$. Reintroducing the units (this can be done uniquely up to a numerical coefficient), we get

$$T_H = \frac{M_{pl}^2 c^2}{8\pi k_b M} = 6 \times 10^{-8}\text{K} \tag{19}$$

The typical wavelength, by Wien's displacement law, is inversely proportional to the temperature

$$\lambda \sim \frac{M_{pl}c^2}{k_b T_H} l_{pl} \sim r_s \approx 3\text{km} \tag{20}$$

which is of the order of the Schwarzschild's radius.

(b) The power radiated by the black hole is

$$P = \sigma T_H^4 \times A = \frac{2\pi^5}{15}\frac{1}{(8\pi)^4}16\pi\frac{M_{pl}^2}{M^2}\frac{M_{pl}c^2}{t_{pl}} = \frac{q}{M^2} \tag{21}$$

where $q \equiv \frac{\pi^2}{1920}\frac{M_{pl}^3 c^2}{t_{pl}}$. To get an idea of the magnitude, a black hole with the same mass as the sun is characterised by $P \approx 10^{-26}$W.
Hence we have

$$\frac{d(Mc^2)}{dt} = -\frac{q}{M^2} \tag{22}$$

$$t_{evap} = -\int_{M_0}^{0} dM\frac{M^2c^2}{q} = \frac{M_0^3c^2}{3q} = \frac{M_0^3}{3M_{pl}^3}t_{pl} \approx 10^{65}\text{years} \tag{23}$$

The power emitted by black holes is very small compared to that of any star (e.g. $\mathcal{O}(10^{26}W)$ for the sun, and it is even lower for large black holes. Moreover, star-sized black holes, which are the smallest known to form from gravitational collapse of supernovae, take extremely long times to evaporate. Finally, the radiation temperature is extremely low even compared to the CMB temperature.
(No wonder Hawking did not win a Nobel prize for this...)

(c) Consider the RN horizons at radii $r_\pm = M \pm \sqrt{M^2 - Q^2}$.
The black hole entropy is $S_{BH} = \pi r_+^2$, while the temperature is (see previous problem set) $T_H = \frac{r_+ - r_-}{4\pi r_+^2} = \frac{2r_+ - 2M}{4\pi r_+^2}$.
After some straightforward computations, we get:

$$C = T_H \frac{\left.\frac{dS_{BH}}{dr_+}\right|_Q}{\left.\frac{dT_H}{dr_+}\right|_Q} = \frac{2S_{BH}\sqrt{M^2 - Q^2}}{M - 2\sqrt{M^2 - Q^2}} \tag{24}$$

(d) Suppose we slightly increase the temperature of the heat reservoir, then energy will flow into the black hole, increasing its mass. If the heat capacity of the black hole is negative, and therefore as the black hole gains mass, it will decrease in temperature, causing more flux from the reservoir to the black hole. Thus there is a positive feedback effect; the system is unstable.

Now consider 24: the numerator of the heat capacity is always positive, while the denominator changes sign when $M = M_{cr} = \frac{2}{\sqrt{3}}|Q|$.
When $M < M_{cr}$, $C > 0$, hence the black hole is stable. Note that Schwarzschild black holes can never be stable.